

A k-Inflated Negative Binomial Mixture Regression Model: Application to Rate-Making Systems

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Abstract

This article introduces a k-Inflated Negative Binomial mixture distribution/regression model as a more flexible alternative to zero-inflated Poisson distribution/regression model. An EM algorithm has been employed to estimate the model's parameters. Then, such new model along with a Pareto mixture model have been employed to design an optimal rate-making system. Namely, this article employs number/size of reported claims of Iranian third party insurance dataset. Then, it employs the k-Inflated Negative Binomial mixture distribution/regression model as well as other well developed counting models along with a Pareto mixture model to model frequency/severity of reported claims in Iranian third party insurance dataset. Such numerical illustration shows that: **(1)** the k-Inflated Negative Binomial mixture models provide more fair rate/pure premiums for policyholders under a rate-making system; and **(2)** in the situation that number of reported claims uniformly distributed in past experience of a policyholder (for instance $k_1 = 1$ and $k_2 = 1$ instead of $k_1 = 0$ and $k_2 = 2$). The rate/pure premium under the k-Inflated Negative Binomial mixture models are more appealing and acceptable.

Keywords: Negative Binomial regression; Poisson regression; Mixture model; Overdispersed behavior; Heavy-tail behavior; Inflated model; EM algorithm; Rate-making System.

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1. Introduction

Modeling count data is an interesting topic in variety fields of applied sciences, such as actuarial sciences, economics, sociology, engineering, etc. In many practical situation the popular classical Poisson regression model fails to model count data which exhibit overdispersion (i.e., the variance of the response variable exceeds its mean). Moreover, strict assumptions on Poisson distribution make it more less applicable in situation that such assumption cannot be strictly verified. The Negative Binomial distribution/regression has become more and more popular as a more flexible alternative to Poisson distribution/regression. In a situation that strict requirements for Poisson distribution cannot be verified, the Negative Binomial distribution is an appropriate choice (Johnson et al., 2005). Moreover, the Negative Binomial is an appropriate choice for overdispersed count data that are not necessarily heavy-tailed (Aryuyuen & Bodhisuwan, 2013).

For count data, the overdispersed behavior has been arrived by *either* observing excess of a single value more than number of expected under the model *or* the target population consisting of several sub-populations. Using k-Inflated and mixture models are two popular statistical approach to dealing with an overdispersed behavior. Simar (1976) and Laird (1978) were two authors who employed Poisson mixture models to considering an overdispersed behavior. Lambert (1992) considered a zero-inflated Poisson regression model to take into account an overdispersed behavior. Wedel et al. (1993) Brännäas & Rosenqvist (1994), Wang et al. (1996), Alfó & Trovato (2004), Wang et al. (1998), among others, developed idea of using a finite mixture Poisson regression model to handel overdispersion.

Greene (1994) and Hall (2000) were pioneer authors who employed zero-inflated Negative Binomial regression to model overdispersion. The ordinary Negative Binomial distribution can be viewed as a mixture of Poisson and gamma distributions (Simon, 1961). To handel an overdispersion phe-

nomena, several extension of Negative Binomial distribution have been introduced by authors. For instance Negative Binomial exponential distribution (Panjer & Willmot, 1981), Negative Binomial Pareto distribution (Meng et al., 1999), Negative Binomial Inverse Gaussian distribution (Gómez-Déniz et al., 2008), Negative Binomial Lindley distribution (Zamani & Ismail, 2010), Negative Binomial Beta Exponential distribution (Pudprommarat, 2012), and Negative Binomial Generalized Exponential distribution (Pudprommarat et al., 2012).

In 2014, Lim et al. considered a k-Inflated Poisson mixture model which simultaneously takes into account both inflated and mixture approaches to handel an overdispersion phenomena. Moreover Tzougas et al. (2014)' introduced a Negative Binomial mixture model to model an overdispersion phenomena. This article follows Lim et al. (2014)'s and generalized Tzougas et al. (2014)'s findings. More precisely, It introduces a k-Inflated Negative Binomial mixture distribution/regression. To show practical application of our finding, we consider the problem of designing an optimal rate-making system. Then, premium of such optimal rate-making system has been evaluated using the result of this article.

This article has been structured as follows. The k-Inflated Negative Binomial mixture model, some of its properties, and an EM algorithm, to estimated its parameters, have been developed in Section 2. The Pareto mixture regression model has been given in Section 3. Application of the k-Inflated Negative Binomial mixture model along with a Pareto mixture model to design an optimal rate-making system have been given in Section 4. Section 5 employs our model, as well as other well-known model, to evaluate rate, base, and pure premiums under a rate-making system for Iranian third party insurance dataset. Base upon three comparison methods, Section 6 shows that our model provides more accurate (in some sense) results. Section 7 employs our well fitted models to calculate rate and pure premiums under two different Scenarios. Conclusion and suggestions have been given in Section 8

2. k-Inflated Negative Binomial mixture regression model

The k-Inflated Negative Binomial mixture, say kINBM, distribution arrives by combining m weighted mixture Negative Binomial distribution with a single mass at point k . The probability mass function for a kINBM distribution has been given by

$$P(Y = y|\boldsymbol{\theta}) = p_1 I_{(y=k)} + \sum_{j=2}^m p_j \binom{y + \alpha_j - 1}{y} \left(\frac{\tau_j}{\alpha_j + \tau_j} \right)^{\alpha_j} \left(\frac{\alpha_j}{\alpha_j + \tau_j} \right)^y I_{\mathbb{N}}(y), \quad (1)$$

where $k \in \mathbb{N}$ and $\boldsymbol{\theta}$ stands for all $3m$ unknown parameters. Moreover, $\sum_{j=1}^m p_j = 1$ and $p_j, \alpha_j, \tau_j \geq 0$, for all $j = 1, \dots, m$. By a straightforward calculation, one may show that

$$\begin{aligned} E(Y) &= p_1 k + \sum_{j=2}^m p_j \alpha_j^2 / \tau_j \\ M_Y(t) &= p_1 e^{kt} + \sum_{j=2}^m p_j \left(\frac{\tau_j}{\tau_j + \alpha_j(1 - e^t)} \right)^{\alpha_j}, t \leq -\max \left\{ \log\left(\frac{\alpha_j}{\alpha_j + \tau_j}\right), j = 2, \dots, m \right\} \\ F_Y(r) &= p_1 I_{[k, \infty)}(r) + 1 - p_1 + \sum_{j=2}^m p_j RIBeta_{\alpha_j/(\alpha_j + \tau_j)}(r + 1; \alpha_j), \end{aligned}$$

where $RIBeta_x(a; b) = \int_0^x t^{a-1} (1-t)^{b-1} dt \Gamma(a+b) / (\Gamma(a)\Gamma(b))$, for $a, b \geq 0, x \in [0, 1]$, stands for the regularized incomplete beta function.

It is well-known that a Negative Binomial distribution can be arrived by mixing two Poisson and gamma distributions (Simon, 1961). The following generalized the above fact to the kINBM distribution.

Corollary 1. *Suppose random variable Y , given parameter $\Lambda = \lambda$, has been distributed according to a k-Inflated Poisson distribution with probability mass function $P(Y = y|\lambda) = p I_{\{k\}}(y) + q \exp(-\lambda_i) (\lambda_i)^y / y!$, where p & $q \in [0, 1]$ and $p + q = 1$. Moreover, suppose that parameter λ has been distributed according to a finite mixture gamma distribution $f_{\Lambda}(\lambda) = \sum_{j=1}^m \varphi_j \lambda^{\alpha_j-1} \tau_j^{\alpha_j} e^{-\tau_j \lambda} / \Gamma(\alpha_j)$, where, for all $j = 1, \dots, m$, $\varphi_j \in [0, 1]$ and $\sum_{j=1}^m \varphi_j = 1$. Then, unconditional distribution of Y has a kINB finite mixture distribution with probability mass function*

$$P(Y = y) = p I_{\{k\}}(y) + q \sum_{j=1}^m \varphi_j \binom{y + \alpha_j - 1}{y} \left(\frac{\tau_j}{1 + \tau_j} \right)^{\alpha_j} \left(\frac{1}{1 + \tau_j} \right)^y. \quad (2)$$

For practical application, in Equation (2), we set $q := \sum_{s=1}^m \omega_s$ and $\varphi_j := \omega_j / \sum_{s=1}^m \omega_s$.

Now, to formulated a kINBM regression model, suppose that for an i^{th} individual, information on count response variables Y_{i1}, \dots, Y_{it} along with information on p covariates X_1, \dots, X_p are available. Also suppose that Y_{il} given parameter $\Lambda_{il} = \lambda_{il}$ has been distributed according to a k-Inflated Poisson distribution with probability mass function $P(Y_{il} = y_{il} | \lambda_{il}) = pI_{\{k\}}(y_{il}) + q \exp(-\lambda_{il})(\lambda_{il})_{y_{il}}^y / y_{il}!$, where $p \& q \in [0, 1]$ and $p + q = 1$. Moreover, suppose that parameter λ_{il} can be evaluated by the following regression model

$$\log(\lambda_{il}) = \beta_{0il} + \sum_{k=1}^p \beta_{kil} x_{ki} + \epsilon_i,$$

where $\beta_{0i}, \dots, \beta_{pi}$ are regression coefficients and $u_i = \exp(\epsilon_i)$ has been distributed according to a finite mixture gamma distribution with density function

$$f_{U_i}(u_i) = \sum_{j=2}^m \varphi_j \frac{u_i^{\alpha_j-1} \alpha_j^{\alpha_j} e^{-\alpha_j u_i}}{\Gamma(\alpha_j)}, \quad (3)$$

where $\sum_{j=2}^m \varphi_j = 1$ and $\alpha_j \geq 0$. To have $E(\epsilon_i) = 0$, we set both parameters of all gamma distributions, in the finite mixture gamma distribution, to be equal.

Using the law of total probability and setting $d_{il} := \beta_{0il} + \sum_{k=1}^p \beta_{kil} x_{ki}$, one may show that

$$\begin{aligned} P(Y_{il} = y_{il} | \theta) &= \int_0^\infty P(Y_{il} = y_{il} | \theta, u_i) f_{U_i}(u_i) du_i \\ &= pI_{\{k\}}(y_{il}) + \sum_{j=2}^m q\varphi_j \frac{d_{ijl}^{y_{il}} \alpha_j^{\alpha_j}}{y_{il}! \Gamma(\alpha_j)} \int_0^\infty e^{-(d_{ijl} + \alpha_j)u_i} u_i^{y_{il} + \alpha_j - 1} du_i \\ &= pI_{\{k\}}(y_{il}) + \sum_{j=2}^m q\varphi_j \frac{\Gamma(y_{il} + \alpha_j)}{y_{il}! \Gamma(\alpha_j)} \frac{d_{ijl}^{y_{il}} \alpha_j^{\alpha_j}}{(d_{ijl} + \alpha_j)^{y_{il} + \alpha_j}}, \end{aligned}$$

where θ stands for all unknown parameters. Now by setting $p = 1/(1 + \sum_{s=2}^m e^{\omega_s})$, $q\varphi_j = e^{\omega_j}/(1 + \sum_{s=2}^m e^{\omega_s})$, for $j = 2, \dots, m$, and $\left(\frac{y_{il} + \alpha_j - 1}{y_{il}}\right) := \frac{\Gamma(y_{il} + \alpha_j)}{y_{il}! \Gamma(\alpha_j)}$, the kINBM regression model can be restated

as

$$\begin{aligned}
P(Y_{il} = y_{il}) &= \frac{1}{1 + \sum_{j=2}^m e^{\omega_j}} I_{\{k\}}(y_{il}) \\
&+ \sum_{j=2}^m \frac{e^{\omega_j}}{1 + \sum_{s=2}^m e^{\omega_s}} \binom{y_{il} + \alpha_j - 1}{y_{il}} t_{ilj}^{\alpha_j} (1 - t_{ilj})^{y_{il}},
\end{aligned} \tag{4}$$

where $t_{ilj} := \alpha_j / (\alpha_j + \exp\{X_i B_{il}\})$ and $X_i B_{il} := \beta_{0il} + \sum_{k=1}^p \beta_{kil} x_{ik}$.

Parameters estimation

All unknown parameters of the kINBM regression (4) can be represented as $\boldsymbol{\theta} := (\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{B})$. Now to provide a Maximum likelihood estimator, say MLE, for $\boldsymbol{\theta}$, one may employ an EM algorithm. In statistical literature, the EM algorithm is a well-known and practical method to obtain the Maximum likelihood estimators for parameters in an arbitrary finite mixture model (McLachlan & Krishnan, 1997). Now suppose that number of components, m , is given, and $\mathbf{v}_i = (v_{i1}, \dots, v_{im})$ stands for the latent vector of component indicator variables, where for $i = 1, \dots, n$ and $j = 1, \dots, m$, $v_{ij} = 1$ whenever observation i comes from j^{th} component and $v_{ij} = 0$, otherwise. Therefore, we assume that each observation has been arrived from one of the m components, but the component it belongs to is unobservable and therefore considered to be the missing data.

Now using the Multinomial distribution for the unobservable vector \mathbf{v}_i , the complete data log-likelihood function, for, the kINB regression model, can be written as the following, see Rigby & Stasinopoulos (2009) for an update information.

$$\begin{aligned}
l_c(\boldsymbol{\theta} | y_i, \mathbf{v}_i, X_i) &= \sum_{i=1}^n v_{i1} \log \left(\frac{1}{1 + \sum_{l=2}^m \exp(\omega_l)} \right) I_{(y_i=k)} \\
&+ \sum_{i=1}^n \sum_{j=2}^m v_{ij} \log \left(\frac{\exp(\omega_j)}{1 + \sum_{l=2}^m \exp(\omega_l)} \binom{y_i + \alpha_j - 1}{y_i} t_j^{\alpha_j} (1 - t_j)^{y_i} \right),
\end{aligned} \tag{5}$$

where $\boldsymbol{\theta} := (\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{B})$ stands for all unknown parameters, $t_j := \alpha_j / (\alpha_j + \exp\{X_i B_j\})$, and $X_i B_j := \beta_{0j} + \sum_{k=1}^p \beta_{kj} x_{ik}$.

The EM algorithm employs the following two steps to maximize the above loglikelihood function.

E-step: In this step, using given data along with current estimates $\hat{\boldsymbol{\theta}}^{(r)} := (\hat{\boldsymbol{\omega}}^{(r)}, \hat{\boldsymbol{\alpha}}^{(r)}, \hat{\boldsymbol{B}}^{(r)})$ obtained from the r^{th} iteration, the probability \hat{v}_{ij} estimates. This probability, $(r+1)^{\text{th}}$ iteration, can be stated as

$$\begin{aligned}\hat{v}_{ij}^{(r+1)} &= E \left[v_{ij} | y_i, X_i, \hat{\boldsymbol{\theta}}^{(r)} \right] = 1 \times P \left(v_{ij} = 1 | y_i, X_i, \hat{\boldsymbol{\theta}}^{(r)} \right) + 0 \times P \left(v_{ij} = 0 | y_i, X_i, \hat{\boldsymbol{\theta}}^{(r)} \right) \\ &= \frac{f \left(y_i | v_{in} = 1, X_i, \hat{\boldsymbol{\theta}}^{(r)} \right) P \left(v_{in} = 1 | X_i, \hat{\boldsymbol{\theta}}^{(r)} \right)}{f \left(y_i | X_i, \hat{\boldsymbol{\theta}}^{(r)} \right)} \\ &= \frac{\exp \left(\hat{\omega}_j^{(r)} \right) \binom{y_i + \hat{\alpha}_j^{(r)} - 1}{y_i} \left(\hat{t}_j^{(r)} \right)^{\hat{\alpha}_j^{(r)}} \left(1 - \hat{t}_j^{(r)} \right)^{y_i}}{1 + \sum_{s=2}^m \exp \left(\hat{\omega}_s^{(r)} \right) \binom{y_i + \hat{\alpha}_s^{(r)} - 1}{y_i} \left(\hat{t}_s^{(r)} \right)^{\hat{\alpha}_s^{(r)}} \left(1 - \hat{t}_s^{(r)} \right)^{y_i}},\end{aligned}$$

where $\hat{t}_j^{(r)} := \hat{\alpha}_j^{(r)} / (\hat{\alpha}_j^{(r)} + \exp\{X_i \hat{B}_j^{(r)}\})$, and $X_i \hat{B}_j^{(r)} := \hat{\beta}_{0j}^{(r)} + \sum_{k=1}^p \hat{\beta}_{kj}^{(r)} x_{ik}$.

M-step: Given the probability \hat{v}_{ij} , this step maximizes, in the $(r+1)^{\text{th}}$ iteration, the following loglikelihood $Q(\cdot)$ with respect to $\boldsymbol{\theta} := (\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{B})$.

$$\begin{aligned}Q &= E \left[l_c | y_i, X_i, \hat{\boldsymbol{\theta}}^{(r)} \right] \\ &= \sum_{i=1}^n \hat{v}_{i1}^{(r+1)} \log \left(\frac{1}{1 + \sum_{l=2}^m \exp(\omega_l)} \right) I_{(y_i=k)} + \sum_{i=1}^n \sum_{l=2}^m \left[\hat{v}_{il}^{(r+1)} \log \left(\frac{\exp(\hat{\omega}_l^{(r)})}{1 + \sum_{s=2}^m \exp(\hat{\omega}_s^{(r)})} \right) \right] \\ &\quad + \sum_{i=1}^n \sum_{l=2}^m \hat{v}_{il}^{(r+1)} \log \left(\binom{y_i + \hat{\alpha}_l^{(r)} - 1}{y_i} \left(\frac{\hat{\alpha}_l^{(r)}}{\hat{\alpha}_l^{(r)} + \exp(X_i \hat{B}_l^{(r)})} \right)^{\hat{\alpha}_l^{(r)}} \left(\frac{\exp(X_i \hat{B}_l^{(r)})}{\hat{\alpha}_l^{(r)} + \exp(X_i \hat{B}_l^{(r)})} \right)^{y_i} \right) \\ &=: Q_1 + Q_2.\end{aligned}$$

Updated parameters $\hat{\boldsymbol{\theta}}^{(r+1)} := (\hat{\boldsymbol{\omega}}^{(r+1)}, \hat{\boldsymbol{\alpha}}^{(r+1)}, \hat{\boldsymbol{B}}^{(r+1)})$ have been arrived by solving the following equation.

$$\begin{aligned}\frac{\partial Q_1}{\partial \omega_j} &= \sum_{i=1}^n \frac{\partial}{\partial \omega_j} \hat{v}_{i1}^{(r+1)} \log \left(\frac{1}{1 + \sum_{l=2}^m \exp(\omega_l)} \right) I_{(y_i=k)} + \sum_{i=1}^n \frac{\partial}{\partial \omega_j} \hat{v}_{ij}^{(r+1)} \log \left(\frac{\exp(\omega_j)}{1 + \sum_{l=2}^m \exp(\omega_l)} \right) = 0; \\ \frac{\partial Q_2}{\partial B_j} &= \sum_{i=1}^n \frac{\partial}{\partial B_j} \hat{v}_{in}^{(r+1)} \log \left(\binom{y_i + \alpha_j - 1}{y_i} \left(\frac{\alpha_j}{\alpha_j + \exp(X_i B_j)} \right)^{\alpha_j} \left(\frac{\exp(X_i B_j)}{\alpha_j + \exp(X_i B_j)} \right)^{y_i} \right) = 0; \\ \frac{\partial Q_2}{\partial \alpha_j} &= \sum_{i=1}^n \frac{\partial}{\partial \alpha_j} \hat{v}_{in}^{(r+1)} \log \left(\binom{y_i + \alpha_j - 1}{y_i} \left(\frac{\alpha_j}{\alpha_j + \exp(X_i B_j)} \right)^{\alpha_j} \left(\frac{\exp(X_i B_j)}{\alpha_j + \exp(X_i B_j)} \right)^{y_i} \right) = 0.\end{aligned}$$

Since the above three equations cannot solve explicitly, such updated parameters have been

obtained using the following Iteratively Reweighted Least Squares, say IRLS, method.

$$\begin{aligned}\hat{\omega}_j^{(r+1)} &= \hat{\omega}_j^{(r)} + \left(E \left(\frac{-\partial^2 Q_1}{\partial \omega_j^2} \right) \right)^{-1} \cdot \frac{\partial Q_1}{\partial \omega_j}; \\ \hat{B}_j^{(r+1)} &= \hat{B}_j^{(r)} + \left(E \left(\frac{-\partial^2 Q_2}{\partial B_j^2} \right) \right)^{-1} \cdot \frac{\partial Q_2}{\partial B_j}; \\ \hat{\alpha}_j^{(r+1)} &= \hat{\alpha}_j^{(r)} + \left(E \left(\frac{-\partial^2 Q_2}{\partial \alpha_j^2} \right) \right)^{-1} \cdot \frac{\partial Q_2}{\partial \alpha_j}.\end{aligned}$$

In IRLS method, $E(-\partial^2(\text{loglikelihood})/\partial \text{parameter}^2)$ can be viewed as the Fisher information matrix and $\partial(\text{loglikelihood})/\partial \text{parameter}$ as score function.

After updated Parameter estimates $\hat{\boldsymbol{\theta}}^{(r+1)} := (\hat{\boldsymbol{\omega}}^{(r+1)}, \hat{\boldsymbol{\alpha}}^{(r+1)}, \hat{\boldsymbol{B}}^{(r+1)})$, the complete data loglikelihood for $(r+1)^{\text{th}}$ iteration, arrives by

$$\begin{aligned}l_c^{(r+1)} &= \sum_{i=1}^n \hat{v}_{i1}^{(r+1)} \log \left(\frac{1}{1 + \sum_{l=2}^m \exp(\hat{\omega}_l^{(r+1)})} \right) I_{(y_i=k)} \\ &\quad + \sum_{i=1}^n \sum_{j=2}^m \hat{v}_{ij}^{(r+1)} \log \left(\frac{\exp(\hat{\omega}_j^{(r+1)})}{1 + \sum_{l=2}^m \exp(\hat{\omega}_l^{(r+1)})} \left(\frac{y_i + \hat{\alpha}_j^{(r+1)}}{y_i} - 1 \right) (\hat{\mathbf{t}}_j^{(r+1)})^{\hat{\alpha}_j^{(r+1)}} (1 - \hat{\mathbf{t}}_j^{(r+1)})^{y_i} \right),\end{aligned}$$

where $\hat{\mathbf{t}}_j^{(r+1)} := \hat{\alpha}_j^{(r+1)} / (\hat{\alpha}_j^{(r+1)} + \exp\{X_i \hat{B}_j^{(r+1)}\})$, and $X_i \hat{B}_j^{(r+1)} := \hat{\beta}_{0j}^{(r+1)} + \sum_{k=1}^p \hat{\beta}_{kj}^{(r+1)} x_{ik}$. Now, in the E-step v_{ij} -s have been estimated. This loop has been repeated until the difference $|l_c^{(r+1)} - l_c^{(r)}|$ has been converged, in some sense.

It is worthwhile to mention that, since regression coefficients $\mathbf{B} = (\beta_0, \dots, \beta_p)'$ have been estimated using the MLE methods. therefore, number of mixture component impact on such estimators.

3. Pareto mixture regression model

The Pareto mixture distribution arrives by combining m weighted mixture Pareto distributions. The density function for a Pareto mixture distribution has been given by

$$f_Z(z|\boldsymbol{\vartheta}) = \sum_{j=1}^m \rho_j \alpha_j \frac{\gamma_j^{\alpha_j}}{(z + \gamma_j)^{\alpha_j+1}}, \quad (6)$$

where $\boldsymbol{\vartheta} = (\boldsymbol{\rho}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ stands for all $3m$ unknown parameters. Moreover, $\sum_{j=1}^m \rho_j = 1$ and $\rho_j, \alpha_j \geq 0$, for all $j = 1, \dots, m$. More details on this distribution can be found in Tzougas et al. (2014).

Tzougas et al. (2014) showed that a Pareto mixture distribution can be arrived by mixing two exponential and inverse gamma distributions.

Now, to formulated a Pareto mixture regression model, suppose that for an i^{th} individual, information on continuous response variables Z_{i1}, \dots, Z_{it} along with information on p covariates W_1, \dots, W_p are available. Also suppose that Z_{il} given parameter $\Theta_{il} = \theta_{il}$ has been distributed according to an exponential distribution with density function $f_{Z_{il}|\Theta_{il}=\theta_{il}}(z_{il}) = \exp\{-z_{il}/\theta_{il}\}/\theta_{il}$. Moreover, suppose that parameter θ_{il} can be evaluated by the following regression model

$$\log(\theta_{il}) = d_{0il} + \sum_{k=1}^p d_{kil}w_{ki} + \epsilon_i,$$

where d_{0i}, \dots, d_{pi} are regression coefficients and $u_i = \exp(\epsilon_i)$ has been distributed according to a finite mixture Inverse gamma distribution with density function

$$f_{U_i}(u_i) = \sum_{j=1}^m \rho_j \frac{(\alpha_j - 1)^{\alpha_j} u_i^{-\alpha_j - 1}}{\Gamma(\alpha_j)} e^{-(\alpha_j - 1)/u_i}, \quad (7)$$

where $\sum_{j=1}^m \rho_j = 1$ and $\rho_j, \alpha_j \geq 0$. To have $E(\epsilon_i) = 0$ in Equation (6) we set $\gamma_j = \alpha_j - 1$, for $j = 1, \dots, m$.

Using the law of total probability and setting $b_{il} := d_{0il} + \sum_{k=1}^p d_{kil}w_{ki}$, one may show that

$$\begin{aligned} f_{Z_{il}|\vartheta}(z_{il}) &= \int_0^\infty f_{Z_{il}|\vartheta, u_i}(z_{il}) f_{U_i}(u_i) du_i \\ &= \int_0^\infty e^{-z_{il} \exp\{-b_{il}\}/u_i} \exp\{-b_{il}\} u_i^{-1} \sum_{j=1}^m \rho_j \frac{(\alpha_j - 1)^{\alpha_j} u_i^{-\alpha_j - 1}}{\Gamma(\alpha_j)} e^{-(\alpha_j - 1)/u_i} du_i \\ &= \sum_{j=1}^m \rho_j \alpha_j \frac{(\alpha_j - 1)^{\alpha_j}}{(z + \alpha_j - 1)^{\alpha_j + 1}}. \end{aligned}$$

Similar to the kINB regression/distribution the maximum liklihood estimator for parameters of a Pareto mixture regression/distribution can be obtained using the EM algorithm. Fortunately, Rigby & Stasinopoulos (2001) developed a R package, named 'GAMLSS', for such propose, see Rigby & Stasinopoulos (2001, 2009) for more details.

4. Application to posteriori rate-making system

The rate-making system is a non-life actuarial system which rates policyholders based upon their last t years record (Payandeh Najafabadi et al., 2015). A rate-making system based upon policyholders' characteristics assigns a priori premium for each policyholder. Then, it employs the last t years claims experience of each insured to update such priori premium and provides posteriori premium (Boucher & Inoussa, 2014). The Bonus-Malus system is a commercial and practical version of the rate-making system which takes into account current year policyholders' experience to determine their next year premium.

There is a considerable attention from authors to study rate-making systems (or Bonus-Malus systems). For instance: Several mathematical tools for pricing a rate-making system has been provided by Lange (1969). Dionne & Vanasse (1989, 1992) employed available asymmetric information under Poisson and Negative Binomial regression models to determine premium of a rate-making system. In 1995, Lemaire designed an optimal Bonus-Malus system based on Negative Binomial distribution. Pinquet (1997) considered Poisson and Lognormal distributions to design an optimal Bonus-Malus system. Walhin & Paris (1999) considered a Hofmann's distribution along with a finite mixture Poisson distribution to evaluate elements of a Bonus-Malus system. The relatively premium of a rate-making system under the exponential loss function has been evaluated by Denuit & Dhaene (2001). In 2001, Frangos & Vrontos designed an optimal Bonus-Malus system using both Pareto and Negative Binomial distributions. Using the bivariate Poisson regression model Bermúdez & Morata (2009) studied priori rate-making procedure for an automobile insurance database which has two different types of claims. In 2011, Bermúdez & Karlis employed a Bayesian multivariate Poisson model to determine premium of a rate-making system which has a non-ignorable correlation between types of its claims. Boucher & Inoussa (2014) introduced a new model to determine premium of a rate-making system whenever panel or longitudinal data are available. The Sichel distribution along with a Negative Binomial distribution have been considered by Tzougas & Fran-

gos (2013, 2014a, 2014b). Tzougas et al. (2014) employed a finite mixture distribution to model frequency and severity of accidents. Payandeh Najafabadi et al. (2015) employed Payandeh Najafabadi (2010)'s idea to determine credibility premium for a rate-making system whenever number of reported claims distributed according to a zero-inflated Poisson distribution. Several authors have been employed zero-inflated models in actuarial science, see instance Yip & Yau (2005), Boucher et al. (2009), Boucher & Denuit (2008), and Boucher et al. (2007), among others.

Under a rate-making system the pure premium of an i^{th} policyholder at $(t + 1)^{\text{th}}$ year has been estimated by multiplication of estimated base premium, say $\hat{BP}(t+1)$, into corresponding estimated rate premium, say $\hat{Rate}(t + 1)$. From decision theory point of view, the Bayes estimator offers an intellectually and acceptable estimation for both the rate premium $Rate(t+1)$ and the base premium $BP(t + 1)$. Such Bayes estimators, under the quadratic loss function, can be obtained by posterior expectation of risk parameters given number and severity of reported claims at first $t + 1$ years, see Denuit et al. (2007) for more details.

Therefore, to determine premium for i^{th} policyholder, under a rate-making system, one has to determine both Bayes estimators. The following two theorems develop such estimators. Namely, in the first step, it supposes that number of reported claim Y_1, \dots, Y_t , given risk parameter $\Lambda_i = \lambda_i$, has been distributed according to a k-Inflated Poisson distribution and risk parameter Λ_i distributed as a finite mixture Gamma. In the second step, it supposes that claim size random variable Z_1, \dots, Z_t , given risk parameter $\Theta_i = \theta_i$, has been distributed according to an exponential distribution and risk parameter Θ_i distributed as a finite mixture inverse Gamma. Finally, it derives such Bayes estimators for risk parameters θ_i and λ_i .

Theorem 1. *Suppose that for an i^{th} policyholder, number of reported claims in the last t years have been restated as $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{it})$. Also suppose that, for $l = 1, \dots, t$, Y_{il} given parameter $\Lambda_{il} = \lambda_{ik}$ has been distributed according to a k-Inflated Poisson distribution with probability mass function $P(Y_{il} = y_{il} | \lambda_{il}) = pI_{\{k\}}(y_{il}) + q \exp(-\lambda_{il})(\lambda_{il})_{il}^{y_{il}}/y_{il}!$, where p & $q \in [0, 1]$ and $p + q = 1$. Moreover, suppose that risk parameter Λ_i can be restated as regression model $\log(\lambda_{il}) = C_i B_{il} + \epsilon_i$ where $\mathbf{C}_i = (1, c_{i,1}, \dots, c_{i,p})$ is the vector of p characteristics/covariates for an i^{th} pol-*

icyholder, $B_{il} = (\beta_{0il}, \dots, \beta_{pil})'$ is the vector of the regression coefficients, and $u_i = \exp(\epsilon_i)$ has been distributed according to finite mixture gamma distribution with density function $f_{U_i}(u_i) = \sum_{j=1}^m \varphi_j u_i^{\alpha_j-1} \alpha_j e^{-\alpha_j u_i} / \Gamma(\alpha_j)$, where $u_i > 0$, $\alpha_j > 0$ and $\sum_{j=1}^m \varphi_j = 1$. Then, Bayes estimator for the rate premium $\widehat{Rate}_i(t+1)$, of an i^{th} policyholder at $(t+1)^{th}$ year, is given by

$$\widehat{Rate}_i(t+1) = e^{C_i B_{it+1}} \frac{\int_0^\infty u_i \prod_{l=1}^t h_{il}(u_i) \sum_{j=1}^m k_j(u_i) du_i}{\int_0^\infty \prod_{l=1}^t h_{il}(u_i) \sum_{j=1}^m k_j(u_i) du_i}, \quad (8)$$

where $k_j(u_i) := \varphi_j u_i^{\alpha_j-1} \alpha_j e^{-\alpha_j u_i} / \Gamma(\alpha_j)$, $\Gamma(\cdot)$ stands for the Gamma function, and $h_{il}(u_i) := pI_{(y_{il}=k)} + qe^{-\exp(C_i B_{il})u_i} (\exp(C_i B_{il})u_i)^{y_{il}} / y_{il}!$.

Proof. The Bayes estimator for the rate premium $Rate_i(t+1)$, under the quadratic loss function, is mean of posterior distribution $\Lambda_{it+1} | (\mathbf{Y}_i, \mathbf{C}_i)$. Such the posterior distribution can be restated as the following.

$$\begin{aligned} f_{\Lambda_{it+1} | (\mathbf{Y}_i, \mathbf{C}_i)}(\lambda_{it+1}) &= \frac{\prod_{l=1}^t P(Y_{il} = y_{il} | \Lambda_{il}) P(\Lambda_{il} = e^{C_i B_{il}} u_i)}{\int_0^\infty \prod_{l=1}^t P(Y_{il} = y_{il} | \Lambda_{il}) P(\Lambda_{il} = e^{C_i B_{il}} u_i) du_i} \\ &= \frac{\prod_{l=1}^t h_{il}(u_i) \sum_{j=1}^m k_j(u_i)}{\int_0^\infty \prod_{l=1}^t h_{il}(u_i) \sum_{j=1}^m k_j(u_i) du_i}. \end{aligned}$$

Now the desired result arrives by

$$\widehat{Rate}_i(t+1) = \int_0^\infty e^{C_i B_{it+1}} u_i f_{\Lambda_{it+1} | (\mathbf{Y}_i, \mathbf{C}_i)}(\lambda_{it+1}) du_i \quad \square$$

In a situation that $q = 1$, the rate premium $\hat{Rate}_i(t+1)$ can be restated as

$$\widehat{Rate}_i(t+1) = e^{C_i B_{it+1}} \frac{\sum_{j=1}^m \varphi_j \alpha_j^{\alpha_j} \Gamma(\alpha_j + y_{i\cdot} + 1) / (\Gamma(\alpha_j)(\alpha_j + \sum_{l=1}^t e^{C_i B_{il}}))}{\sum_{j=1}^m \varphi_j \alpha_j^{\alpha_j} \Gamma(\alpha_j + y_{i\cdot}) / (\Gamma(\alpha_j)(\alpha_j + \sum_{l=1}^t e^{C_i B_{il}}))},$$

where $y_{i\cdot} = \sum_{l=1}^t y_{il}$. This situation has been studied by Dionne & Vanasse (1992) for an one mixture distribution and by Tzougas et al. (2014) for an m mixture distribution. In the case that $t = 0$, one may show that $\widehat{Rate}_i(1) = e^{C_i B_{i1}}$.

Remark 1. For the situation that no covariate information has been taken into account, say a distribution model, and the risk parameter Λ_i has been distributed according to a finite mixture gamma distribution with density function given by (3). Result of Theorem (1) can be reformulated as

$$\widehat{Rate}_i(t+1) = \frac{\int_0^\infty \lambda_{it+1} \prod_{l=1}^t \left(pI_{(y_{il}=k)} + q \frac{e^{-\lambda_{it+1}} (\lambda_{it+1})^{y_{il}}}{y_{il}!} \right) \sum_{j=1}^m \varphi_j \frac{\lambda_{it+1}^{\alpha_j-1} \tau_j^{\alpha_j} e^{-\tau_j \lambda_{it+1}}}{\Gamma(\alpha_j)} d\lambda_{it+1}}{\int_0^\infty \prod_{l=1}^t \left(pI_{(y_{il}=k)} + q \frac{e^{-\lambda_{it+1}} (\lambda_{it+1})^{y_{il}}}{y_{il}!} \right) \sum_{j=1}^m \varphi_j \frac{\lambda_{it+1}^{\alpha_j-1} \tau_j^{\alpha_j} e^{-\tau_j \lambda_{it+1}}}{\Gamma(\alpha_j)} d\lambda_{it+1}}.$$

The following theorem develops a Bayes estimator for the base premium $BP_i(t+1)$ for an i^{th} policyholder at $(t+1)^{\text{th}}$ year.

Theorem 2. Suppose that for an i^{th} policyholder, severity/size of claims in the last t years have been restated as $\mathbf{Z}_i = (\mathbf{Z}_{i1}, \dots, \mathbf{Z}_{it})$. Also suppose that, for $l = 1, \dots, t$, $\mathbf{Z}_{il} = (Z_{il1}, \dots, Z_{ilk_{il}})$, where k_{il} stands for number of reported claims by i^{th} policyholder at l^{th} year, and for $s = 1, \dots, k_{il}$, assume that Z_{ils} given parameter $\Theta_{il} = \theta_{il}$ has been distributed according to an exponential distribution function with density function $f_{Z_{il}|\Theta_{il}=\theta_{il}}(z_{il}) = \exp\{-z_{il}/\theta_{il}\}/\theta_{il}$. Moreover, suppose that risk parameter Θ_i can be restated as $\log(\theta_{i,l}) = \mathbf{W}_i \mathbf{D}_{il} + \epsilon_i$, where $\mathbf{W}_i = (1, w_{i,1}, \dots, w_{i,p})$ is the vector of p characteristics/covariates for an i^{th} policyholder, $\mathbf{D}_{il} = (d_{0il}, \dots, d_{pil})'$ is the vector of the regression coefficients, and $u_i = \exp(\epsilon_i)$ has been distributed according to a finite mixture Inverse Gamma with density function $f_{U_i}(u_i) = \sum_{j=1}^m \phi_j (\eta_j - 1)^{\eta_j} u_i^{-\eta_j-1} \exp(-(\eta_j - 1)/u_i) / \Gamma(\eta_j)$, where $u_i > 0$, $\eta_j > 0$ and $\sum_{j=1}^m \phi_j = 1$. Then, Bayes estimator for the the base premium $BP_i(t+1)$ for an i^{th} policyholder at $(t+1)^{\text{th}}$ year, is given by

$$\widehat{BP}_i^{t+1} = e^{W_i D_{it+1}} \frac{\sum_{j=1}^m \phi_j \frac{(\eta_j-1)^{\eta_j}}{\Gamma(\eta_j)} \frac{\Gamma(\eta_j+K_i-1)}{(\eta_j+\sum_{l=1}^t \sum_{s=1}^{k_{il}} \exp(-W_i D_{il}) z_{ils}-1)^{\eta_j+K_i-1}}}{\sum_{j=1}^m \phi_j \frac{(\eta_j-1)^{\eta_j}}{\Gamma(\eta_j)} \frac{\Gamma(\eta_j+K_i)}{(\eta_j+\sum_{l=1}^t \sum_{s=1}^{k_{il}} \exp(-W_i D_{il}) z_{ils}-1)^{\eta_j+K_i}}}, \quad (9)$$

where $K_i = \sum_{l=1}^t k_{il}$.

Proof. The posterior distribution of $\Theta_{it+1} | (\mathbf{Z}_i, \mathbf{W}_i)$ can be restated as

$$\begin{aligned} f_{\Theta_{it+1} | (\mathbf{Z}_i, \mathbf{W}_i)}(\theta_{it+1}) &= \frac{\prod_{l=1}^t \prod_{s=1}^{k_{il}} f_{Z_{ils} | \Theta_{il}}(z_{ils}) f_{\Theta_{il}}(e^{W_i D_{il}} u_i)}{\int_0^\infty \prod_{l=1}^t \prod_{s=1}^{k_{il}} f_{Z_{ils} | \Theta_{il}}(z_{ils}) f_{\Theta_{il}}(e^{W_i D_{il}} u_i) du_i} \\ &= \frac{\sum_{j=1}^m \phi_j \frac{(\eta_j-1)^{\eta_j}}{\Gamma(\eta_j)} u_i^{-\eta_j-K_i-1} \exp\left\{-\frac{\eta_j+\sum_{l=1}^t \sum_{s=1}^{k_{il}} \exp(-W_i D_{il}) z_{ils}-1}{u_i}\right\}}{\int_0^\infty \sum_{j=1}^m \phi_j \frac{(\eta_j-1)^{\eta_j}}{\Gamma(\eta_j)} u_i^{-\eta_j-K_i-1} \exp\left\{-\frac{(\eta_j+\sum_{l=1}^t \sum_{s=1}^{k_{il}} \exp(-W_i D_{il}) z_{ils}-1)}{u_i}\right\} du_i}. \end{aligned}$$

The desired Bayes estimator arrives by

$$\widehat{BP}_i^{t+1} = \int_0^\infty e^{W_i D_{it+1}} u_i f_{\Theta_{it+1} | (\mathbf{Z}_i, \mathbf{W}_i)}(\theta_{it+1}) du_i. \quad \square$$

The above result also obtained by Tzougas et al. (2014).

Remark 2. For the situation that no covariate information has been taken into account, say a distribution model, and the risk parameter Θ_i has been distributed according to a finite mixture Inverse Gamma with density function given by (7). Result of Theorem (2) can be reformulated as

$$\widehat{BP}_i^{t+1} = \frac{\sum_{j=1}^m \phi_j \frac{(\eta_j-1)^{\eta_j}}{\Gamma(\eta_j)} \frac{\Gamma(\eta_j+K_i-1)}{(\eta_j+\sum_{l=1}^t \sum_{s=1}^{k_{il}} z_{ils})^{\eta_j+K_i-1}}}{\sum_{j=1}^m \phi_j \frac{(\eta_j-1)^{\eta_j}}{\Gamma(\eta_j)} \frac{\Gamma(\eta_j+K_i)}{(\eta_j+\sum_{l=1}^t \sum_{s=1}^{k_{il}} z_{ils})^{\eta_j+K_i}}}.$$

To show practical application of our findings, the next section provides an real example.

5. Numerical Application

Now, we considered available data from Iranian third party liability, at 2011 year. After a primary investigation, we just trusted information about 8874 policyholders. We used 4 independent variables, as covariates, presented in Table 1. For each policyholder we have the initial information at the beginning of the period and we are interested such covariates to model frequency/severity of claims for evaluating pure premium of each policyholder under a rate-making system.

Table 1: Available covariates information for each policyholder.

Variable	Description
Gender	Equal to 0 for woman & 1 for man
Age	Equal to 1 for $18 \leq age < 30$; 2 for $30 \leq age < 40$; 3 for $40 \leq age < 50$; & 4 for $50 \leq age$
Car's price	Equal to 1 for $price < 2 \times 10^4$; 2 for $2 \times 10^4 \leq price < 5 \times 10^4$; 3 for $5 \times 10^4 \leq price < 10^5$; & 4 for $10^5 \leq price$
Living area	Equal to 1 for $population\ size < 10^5$; 2 for $10^5 \leq population\ size < 5 \times 10^5$; 3 for $5 \times 10^5 \leq population\ size < 10^6$; & 4 for $10^6 \leq population\ size$

For simplicity in presentation hereafter, we represent $kINBM_m$ for a k-Inflated Negative Binomial model with m mixture components and $ParetoM_m$ for a Pareto model with m mixture components.

To find an appropriate distribution for the frequency of claim, in the first step, we considered the $kINBM_m$ model along with all distributions that have been considered, by authors, to model frequency of claims in a rate-making system. Namely, we considered the kINBM, Delaporte, Sichel, and Poisson Inverse Gaussian, say PIG, distributions for frequency and the $ParetoM_m$ distribution for severity and estimate their parameters.

The maximum likelihood estimator for the $kINBM_m$ we develop our R codes while the maximum likelihood estimator for other distributions have been computed using the GAMLSS package in R. Table 2 represents the maximum likelihood estimator for significant parameters of such distributions. The significant test for each parameter has been tested by the Wald test.

Now using a backward elimination selection method, we find covariates that may impact on response variable for each regression model. The significant test for each covariate has been done by the Wald test. Table 3 shows result of the backward selection method for frequency/severity of accidents.

Table 2. Estimation for parameters on various model for frequency/severity of claims.

Distribution:						
NBM_1	$0INBM_1$	$1INBM_1$	$2INBM_1$	$3INBM_1$	NBM_2	$0INBM_2$
$\omega = (*, 1)$	$\omega = (0.001, 0.999)$	$\omega = (0.136, 0.861)$	$\omega = (0, 1)$	$\omega = (0.001, 0.999)$	$\omega = (*, 0.005, 0.995)$	$\omega = (0.001, 0.004, 0.995)$
$\tau = 23.390$	$\tau = 22.256$	$\tau = 1.755$	$\tau = 19.408$	$\tau = 32.333$	$\tau = (14.152, 141.857)$	$\tau = (26.027, 124.000)$
$\alpha = 5.717$	$\alpha = 5.376$	$\alpha = 0.217$	$\alpha = 4.734$	$\alpha = 7.730$	$\alpha = (39.02, 32.53)$	$\alpha = (71.74, 30.31)$
Distribution:						
$1INBM_2$	$2INBM_2$	$3INBM_2$	NBM_3	$0INBM_3$	$1INBM_3$	Delaporte
$\omega = (0.116, 0.831, 0.053)$	$\omega = (0, 1, 0)$	$\omega = (0.001, 0.997, 0.002)$	$\omega = (*, 0.014, 0.982, 0.004)$	$\omega = (0.003, 0.007, 0.986, 0.004)$	$\omega = (0.125, 0.644, 0.033, 0.198)$	$\lambda = 0.243$
$\tau = (8.174, 2.690)$	$\tau = (19.408, 30.250)$	$\tau = (25.316, 42.478)$	$\tau = (124.000, 124.000, 30.250)$	$\tau = (141.857, 141.857, 27.571)$	$\tau = (4.587, 2.623, 4.181)$	$\sigma = 77.67$
$\alpha = (0.807, 2.305)$	$\alpha = (4.735, 6.327)$	$\alpha = (6.054, 8.88)$	$\alpha = (30.47, 29.48, 85.17)$	$\alpha = (33.70, 32.58, 77.67)$	$\alpha = (0.357, 2.628, 0.729)$	$\nu = 0.913$
Distribution:						
Sichel	PIG	Pareto M_1	Pareto M_2	Pareto M_3		
$\mu = 0.242$	$\mu = 0.242$	$\rho = 1$	$\rho = (0.519, 0.481)$	$\rho = (0.332, 0.321, 0.347)$		
$\sigma = NS$	$\sigma = 0.225$	$\alpha = 1.871$	$\alpha = (1.871, 1.871)$	$\alpha = (1.871, 1.873, 1.873)$		
$\nu = -4.961$	—	$\gamma = 16.44$	$\gamma = (16.43, 16.44)$	$\gamma = (16.44, 16.43, 16.43)$		

where the first element in ω stands for weight of inflated part and we use * whenever the distribution is non-inflated distribution and NS stands for not significant at 5% level.

Table 3. Regression coefficients for various model for frequency/severity of claims.

Regression model:							
	NBM_1	$0INBM_1$	$1INBM_1$	$2INBM_1$	$3INBM_1$	NBM_2	$0INBM_2$
	$\omega = (*, 1)$	$\omega = (0.041, 0.959)$	$\omega = (0.111, 0.889)$	$\omega = (0.001, 0.999)$	$\omega = (0.002, 0.998)$	$\omega = (*, 0.893, 0.107)$	$\omega = (0.047, 0.025, 0.928)$
	$\alpha = 7.560$	$\alpha = 22.56$	$\alpha = 0.440$	$\alpha = 7.950$	$\alpha = 9.980$	$\alpha = (0.074, 22.01)$	$\alpha = (0.061, 18.71)$
Intercept	$\beta_0 = -0.738$	$\beta_0 = -0.698$	$\beta_0 = -0.887$	$\beta_0 = -0.744$	$\beta_0 = -0.745$	$\beta_0 = (NS, -0.766)$	$\beta_0 = (NS, -0.719)$
Gender	$\beta_1 = NS$	$\beta_1 = NS$	$\beta_1 = NS$	$\beta_1 = NS$	$\beta_1 = NS$	$\beta_1 = (NS, NS)$	$\beta_1 = (NS, NS)$
Age	$\beta_2 = -0.118$	$\beta_2 = -0.118$	$\beta_2 = -0.213$	$\beta_2 = -0.121$	$\beta_2 = -0.114$	$\beta_2 = (NS, -0.116)$	$\beta_2 = (NS, -0.117)$
Car's price	$\beta_3 = -0.189$	$\beta_3 = -0.189$	$\beta_3 = -0.263$	$\beta_3 = -0.190$	$\beta_3 = -0.196$	$\beta_3 = (0.709, 0.175)$	$\beta_3 = (0.587, 0.180)$
Living area	$\beta_4 = NS$	$\beta_4 = NS$	$\beta_4 = NS$	$\beta_4 = NS$	$\beta_4 = NS$	$\beta_4 = (NS, NS)$	$\beta_4 = (NS, NS)$
Regression model:							
	$1INBM_2$	$2INBM_2$	$3INBM_2$	BM_3	$0INBM_3$	$1INBM_3$	Delaporte
	$\omega = (0.120, 0.833, 0.047)$	$\omega = (0.002, 0.024, 0.974)$	$\omega = (0.002, 0.022, 0.976)$	$\omega = (*, 0.93, 0.02, 0.05)$	$\omega = (0.03, 0.07, 0.46, 0.44)$	$\omega = (0.13, 0.07, 0.40, 0.40)$	$\sigma = 59.56$
	$\alpha = (0.685, 9.471)$	$\alpha = (0.087, 19.00)$	$\alpha = (0.090, 18.94)$	$\alpha = (31.22, 30.95, 0.030)$	$\alpha = (13.29, 11.32, 0.157)$	$\alpha = (9.562, 0.643, 0.396)$	$nu = 0.910$
Intercept	$\beta_0 = (1.009, 0.663)$	$\beta_0 = (NS, -0.775)$	$\beta_0 = (NS, -0.775)$	$\beta_0 = (0.638, 1.032, NS)$	$\beta_0 = (0.531, 0.384, 0.165)$	$\beta_0 = (0.535, 2.514, 2.754)$	$\beta_0 = -0.749$
Gender	$\beta_1 = (-0.185, NS)$	$\beta_1 = (NS, NS)$	$\beta_1 = (NS, NS)$	$\beta_1 = (NS, NS, NS)$	$\beta_1 = (NS, NS, NS)$	$\beta_1 = (0.770, 0.632, NS)$	$\beta_1 = NS$
Age	$\beta_2 = (NS, -0.442)$	$\beta_2 = (NS, -0.115)$	$\beta_2 = (NS, -0.114)$	$\beta_2 = (0.305, 0.337, NS)$	$\beta_2 = (NS, 0.151, 0.165)$	$\beta_2 = (0.343, 0.548, 0.813)$	$\beta_2 = -0.113$
Car's price	$\beta_3 = (0.607, 0.076)$	$\beta_3 = (0.618, 0.177)$	$\beta_3 = (0.675, 0.179)$	$\beta_3 = (NS, 0.676, 0.332)$	$\beta_3 = (0.230, 0.151, 0.310)$	$\beta_3 = (NS, 0.298, 1.545)$	$\beta_3 = -0.189$
Living area	$\beta_4 = (NS, NS)$	$\beta_4 = (NS, NS)$	$\beta_4 = (NS, NS)$	$\beta_4 = (NS, NS, NS)$	$\beta_4 = (NS, NS, NS)$	$\beta_4 = (0.238, 2.382, 0.623)$	$\beta_4 = NS$
Regression model:							
	Sichel	PIG	Pareto M_1	Pareto M_2	Pareto M_3		
	$\sigma = NS$	$\sigma = 0.174$	$\rho = 1$	$\rho = (0.542, 0.458)$	$\rho = (0.341, 0.312, 0.347)$		
	$\nu = -5.688$	—	$\alpha = 1.927$	$\alpha = (1.927, 1.927)$	$\alpha = (1.93, 1.93, 1.93)$		
Intercept	$\beta_0 = -0.737$	$\beta_0 = -0.736$	$\beta_0 = 16.15$	$\beta_0 = (6.15, 6.15)$	$\beta_0 = (16.15, 16.15, 16.15)$		
Gender	$\beta_1 = NS$	$\beta_1 = NS$	$\beta_1 = NS$	$\beta_1 = (NS, NS)$	$\beta_1 = (NS, NS, NS)$		
Age	$\beta_2 = -0.119$	$\beta_2 = -0.119$	$\beta_2 = NS$	$\beta_2 = (NS, NS)$	$\beta_2 = (NS, NS, NS)$		
Car's price	$\beta_3 = -0.190$	$\beta_3 = -0.190$	$\beta_3 = 0.157$	$\beta_3 = (-0.16, -0.16)$	$\beta_3 = (-0.16, -0.16, -0.16)$		
Living area	$\beta_4 = NS$	$\beta_4 = NS$	$\beta_4 = NS$	$\beta_4 = (NS, NS)$	$\beta_4 = (NS, NS, NS)$		

where the first element in ω stands for weight of inflated part and we use * whenever the distribution is non-inflated distribution and NS stands for not significant at 5% level.

5.1. Model comparison

To obtain an appropriate model for a given rate-making system, this section begins by considering the $kINBM_m$ model along with all distributions that have been considered, by authors, to model frequency of claims in a rate-making system. Now in order to compare result of regression/distribution models, we conducted three evaluation approaches. Namely: **(1)** In the first approach, to study performance of count distributions, we employ each fitted distribution, 200 times, to simulate 8874 data. Then, using the mean square error, say MSE, criteria, we compare stimulated data with observed data (see Table 3 for result on such comparison); **(2)** The second approach provides a pairwise comparison between fitted count regression/distribution models based upon *either* the Vuong test (for two non-nested models) *or* the likelihood ratio test (for two nested models), see Table 4 for such comparison study; and finally, **(3)** The third approach employs the Akaike Information Criterion (AIC) and the Schwarz Bayesian information Criterion (SBIC) to compare regression/distribution models for both frequency and severity of claims, result of such comparison has been reported in Table 5.

Generating Data approach:

To study performance of fitted count distributions given in Table 1. We employ the GAMLSS package in R to generate samples from the Delaporte, the Sichel, the Poisson Inverse Gaussian distributions. Lim et al. (2014) introduced an idea to generate sample from a given Zero-inflated Poisson mixture distribution. Now, we employ their idea to generate samples from a given a $kINBM_m$ distribution. Based upon their idea, to generate sample y_i from a $kINBM_m$ distribution with probability mass function

$$P(Y = y|\boldsymbol{\theta}) = p_1 I_{(y=k)} + \sum_{j=2}^m p_j \binom{y + \alpha_j - 1}{y} \left(\frac{\tau_j}{\alpha_j + \tau_j} \right)^{\alpha_j} \left(\frac{\alpha_j}{\alpha_j + \tau_j} \right)^y,$$

where all parameters $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\tau}, \boldsymbol{p})$ and k are given. We start with a dummy variable, say s_i , which generated from an uniform (0,1) distribution. If $0 \leq s_i \leq p_1$, we set $y_i = k$; If $p_1 < s_i \leq p_1 + p_2$, then

y_i is a draw from a Negative Binomial distribution $(\alpha_1, \alpha_1/(\alpha_1 + \tau_1))$; If $p_1 + p_2 < s_i \leq p_1 + p_2 + p_3$, then y_i is a draw from a Negative Binomial distribution $(\alpha_2, \alpha_2/(\alpha_2 + \tau_2))$; If $p_1 + p_2 + p_3 < s_i \leq p_1 + p_2 + p_3 + p_4$, then y_i is a draw from a Negative Binomial distribution $(\alpha_3, \alpha_3/(\alpha_3 + \tau_3))$; and so on.

We employed the GAMLSS package and the above idea to simulate 8874 data (200 times). Table 4 reports mean (mean square error, say MSE) of frequency for such 200 times simulated samples.

Table 4. Mean and the MSE of frequency for generated data under count distributions given in Table 1.

Mean (MSE) of frequency for generated data under Distribution:						
Observed(Freq.)	Delaporte	Sichel	PIG	NBM_1	$0INBM_1$	$1INBM_1$
0(6956)	7018.240(5134.78)	7027.435(6384.40)	7001.370(4548.36)	69980.515(1579.14)	7001.825(2964.21)	6958.575(1498.88)
1(1751)	1614.615(19105.90)	1584.62(28655.82)	1620.145(22066.28)	1626.805(13343.36)	1625.580(16799.97)	1748.300(1598.76)
2(122)	203.210(6774.82)	227.235(11188.30)	224.160(11090.70)	222.960(11472.20)	221.325(11189.41)	120.835(136.98)
3(31)	25.540(55.10)	29.680(26.90)	25.585(49.58)	23.645(81.46)	22.950(73.84)	32.555(33.04)
4(9)	6.470(12.50)	4.230(24.28)	2.485(39.78)	1.910(48.10)	2.145(44.23)	9.510(9.50)
5(3)	2.940(1.82)	0.595(6.12)	0.230(7.78)	0.175(6.94)	0.175(7.90)	2.910(2.54)
6(2)	1.430(1.36)	0.160(3.74)	0.000(4.00)	0.000(4.00)	0.00(4.00)	0.895(1.54)
> 6(0)	1.560(2.88)	0.000(0.00)	0.000(0.00)	0.000(0.00)	0.00(0.00)	0.420(0.53)
Mean (MSE) of frequency for generated data under Distribution:						
Observed(Freq.)	$2INBM_1$	$3INBM_1$	NBM_2	$0INBM_2$	$1INBM_2$	$2INBM_2$
0(6956)	6988.835(2476.51)	7003.555(3652.21)	7019.265(6530.50)	7019.135(3004.08)	6958.730(1529.84)	7002.304(3033.16)
1(1751)	1624.415(15079.51)	1626.055(16892.21)	1619.010(22792.00)	1616.830(6657.27)	1748.690(1448.56)	1621.425(16415.15)
2(122)	232.030(11339.35)	214.305(8663.16)	198.745(6347.56)	200.715(10290.01)	122.380(124.82)	220.400(10608.91)
3(31)	26.405(72.11)	28.205(20.45)	23.780(46.54)	23.940(35.47)	30.530(24.16)	26.855(53.16)
4(9)	2.015(41.56)	1.750(51.70)	7.010(11.76)	7.150(10.84)	9.030(8.68)	2.800(38.85)
5(3)	2.910(2.54)	0.155(8.51)	3.635(2.79)	3.350(3.60)	3.115(2.56)	0.155(9.01)
6(2)	0.115(3.85)	0.210(3.38)	1.730(1.18)	1.640(2.04)	1.075(1.26)	0.095(3.175)
> 6(0)	0.000(0.00)	0.000(0.00)	1.11(3.44)	1.090(2.13)	0.450(0.60)	0.000(0.000)
Mean (MSE) of frequency for generated data under Distribution:						
Observed(Freq.)	$3INBM_2$	BM_3	$0INBM_3$	$1INBM_3$		
0(6956)	7019.200(5955.36)	7015.995(1585.00)	7019.785(7553.77)	6959.035(1127.89)		
1(1751)	1605.105(22133.02)	1619.510(11018.88)	1617.120(21482.31)	1747.870(1038.81)		
2(122)	215.755(7985.44)	200.635(8149.00)	199.745(5686.31)	121.985(213.64)		
3(31)	32.100(56.55)	24.635(65.20)	24.355(124.61)	30.630(52.47)		
4(9)	1.650(54.61)	6.9100(10.76)	6.955(12.64)	9.605(8.34)		
5(3)	0.205(9.04)	3.635(2.78)	3.400(3.20)	3.245(1.94)		
6(2)	0.000(4.00)	1.600(1.14)	1.595(0.675)	1.215(1.37)		
> 6(0)	0.000(0.00)	0.650(1.28)	1.045(2.01)	0.415(0.68)		

Results of the simulation study, given in Table 4, shows that the MSE for the all 1-Inflated Negative Binomial mixture distributions, is considerably less than the MSE of other fitted distributions. Therefore, based upon this simulation study, one may conclude that the 1-Inflated Negative Binomial mixture distributions are appropriate distributions for claim frequency of Iranian policyholders.

The Vuong and the likelihood ratio tests' approach:

To make a decision about statistical hypothesis

H_0 : Observed data came from a population with distribution function F

H_1 : Observed data came from a population with distribution function G .

If both of distributions are belong to a family of distributions with different parameters (nested models), one may employ the likelihood ratio test to make such decision. Otherwise, where models are belong to two different family of distributions (Non-nested models) the Vuong has to used, see Denuit et al. (2007, §2) for more details.

Table 5 represents a pairwise comparison between fitted count regression/distribution models given in Tables 1 and 2.

Table 5. Result of the Vuong test (for two non-nested models) or the likelihood ratio test (for two nested models).			
Panel A: Result of the Vuong test			
Model 1	Model 2	Decision on fitted regression	Decision on fitted distribution
Delaporte	$1INBM_1$	$1INBM_1$ (Statistic=-37.63 & P-value=0.00)	$1INBM_1$ (Statistic=-51.58 & P-value=0.00)
PIG	$1INBM_1$	$1INBM_1$ (Statistic=-33.15 & P-value=0.00)	$1INBM_1$ (Statistic=-38.94 & P-value=0.00)
$0INBM_1$	$1INBM_1$	$1INBM_1$ (Statistic=-24.75 & P-value=0.00)	$1INBM_1$ (Statistic=-35.57 & P-value=0.00)
$2INBM_1$	$1INBM_1$	$1INBM_1$ (Statistic=-26.80 & P-value=0.00)	$1INBM_1$ (Statistic=-37.09 & P-value=0.00)
$3INBM_1$	$1INBM_1$	$1INBM_1$ (Statistic=-22.41 & P-value=0.00)	$1INBM_1$ (Statistic=-31.22 & P-value=0.00)
$0INBM_2$	$1INBM_2$	$1INBM_2$ (Statistic=-43.51 & P-value=0.00)	$1INBM_2$ (Statistic=-52.27 & P-value=0.00)
$2INBM_2$	$1INBM_1$	$1INBM_2$ (Statistic=-40.89 & P-value=0.00)	$1INBM_2$ (Statistic=-37.22 & P-value=0.00)
$3INBM_2$	$1INBM_2$	$1INBM_2$ (Statistic=-40.36 & P-value=0.00)	$1INBM_2$ (Statistic=-33.16 & P-value=0.00)
$0INBM_3$	$1INBM_3$	$1INBM_3$ (Statistic=-34.28 & P-value=0.00)	$1INBM_3$ (Statistic=-20.48 & P-value=0.00)
Panel B: Result of the likelihood ratio test			
Model 1	Model 2	Decision on fitted regression	Decision on fitted distribution
NBM_1	$1INBM_1$	$1INBM_1$ (Statistic=62.21 & P-value=0.00)	$1INBM_1$ (Statistic=105.00 & P-value=0.00)
NBM_2	$1INBM_2$	$1INBM_2$ (Statistic=80.46 & P-value=0.00)	$1INBM_2$ (Statistic=49.66 & P-value=0.00)
NBM_3	$1INBM_3$	$1INBM_3$ (Statistic=111.65 & P-value=0.00)	$1INBM_3$ (Statistic=49.75 & P-value=0.00)
$1INBM_1$	$1INBM_2$	$1INBM_2$ (Statistic=45.35 & P-value=0.00)	$1INBM_1$ (Statistic=0.18 & P-value=0.90)
$1INBM_1$	$1INBM_3$	$1INBM_2$ (Statistic=88.65 & P-value=0.00)	$1INBM_1$ (Statistic=0.21 & P-value=0.87)
$1INBM_2$	$1INBM_3$	$1INBM_3$ (Statistic=42.31 & P-value=0.00)	$1INBM_2$ (Statistic=0.3 & P-value=0.97)

Based upon results of Table 5, one may conclude that the 1-Inflated Negative Binomial mixture distributions/regressions, at 5% significant level, defeat other distributions/regressions.

The Akaike Information Criterion (AIC) and the Schwarz Bayesian information criterion approaches:

The Akaike Information Criterion (AIC) and the Schwarz Bayesian Information Criterion (SBIC) are two measure to select an appropriate model among a set of candidate models. Both criteria are defined based on -2 times the maximum log-likelihood, penalized by *either* number of estimated parameters, for AIC, *or* number of estimated parameters times logarithm of number of observations, for SBIC. Given a set of candidate models, a preferred model is the one which has the minimum AIC (SBIC) value, see Denuit et al. (2007, §1) for more details.

Table 6 provides the AIC and the SBIC for fitted regression/distribution models for both frequency and severity of claims.

Table 6. Result of the Akaike Information Criterion (AIC) and the Schwarz Bayesian information Criterion (SBIC).

Regression model				Distribution model		
Model	df	AIC	SBIC	df	AIC	SBIC
NBM_1	6	10656.42	10698.88	2	10784.70	10798.88
Delaporte	7	10615.42	10665.07	3	10734.99	10756.26
Sichel	7	10648.96	10699.16	3	10772.67	10793.94
PIG	6	10653.90	10696.47	2	10781.11	10795.29
$0INBM_1$	7	10664.59	10714.25	3	10786.74	10808.01
$1INBM_1$	7	10596.11	10645.77	3	10681.69	10702.96
$2INBM_1$	7	10658.36	10708.02	3	10787.05	10808.33
$3INBM_1$	7	10653.08	10702.73	3	10783.25	10804.53
$INBM_2$	13	10635.22	10727.86	5	10735.14	10770.59
$0INBM_2$	14	1064.07	10746.80	6	10737.30	10779.84
$1INBM_2$	14	10558.76	10658.58	6	10687.80	10730.02
$2INBM_2$	14	10638.66	10748.39	6	10793.05	10835.60
$3INBM_2$	14	10632.58	10732.31	6	10789.88	10832.42
NBM_3	20	10632.11	10775.11	8	10741.26	10797.99
$0INBM_3$	21	10654.22	10803.49	9	10743.23	10807.05
$1INBM_3$	21	10536.45	10685.28	9	10693.19	10757.33
Pareto M_1	6	63948.20	63990.75	2	64102.30	64116.48
Pareto M_2	13	63968.20	64054.38	5	64108.30	64143.75
Pareto M_3	20	63974.20	64108.93	8	64114.30	64171.00

The AIC and SBIC for fitted models, given Table 6, show that the 1-Inflated Negative Binomial mixture distributions/regressions are better than other distribution/regression models.

6. Rate-making Examples

To show practical application of our findings. We calculate the rate and pure premiums for the set of well fitted distributions/regression models that were presented in above sections. Since we are interested in the differences between rate premium of various classes. Therefore, we set the rate premium for a new policyholder equal to 1 unite, at $t = 0$. Moreover, we considered three different categories, described in Table 7.

Category	Description
A_1	For a situation that no covariate information have been used for premium calculation
A_2	Whenever, chosen policyholder is a young man at age of 25 years old who owns a car with price greater than 2×10^4 and living in a city with population size larger than 10^6 .
A_3	Whenever, chosen policyholder is a mature woman at the age of 55 years old who owns a car with price less than 2×10^4 and living in a city with population size less than 10^5 .

Now to calculate rate premium for three categories A_1 , A_2 , and A_3 , given in Table 7, using well fitted models. We consider two different approaches. The first approach just considers number of cumulated claims in the last yeas. While the second approach considers the exact number of reported claim for each year in a history of the policyholder. ²

Tables 8 and 9 represent calculated rate premium for three categories, given in Table 7, using well fitted models for both approaches.

Table 8: The rate premium for three categories A_1 , A_2 , and A_3 using well fitted models, whenever number of cumulated claims has been considered.

		Model:								
Number of cumulated		NBM_1			NBM_2			NBM_3		
Year	claims up to this year (K)	A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$t = 0$	—	1	1	1	1	1	1	1	1	1
$t = 1$	$K = 0$	0.96	0.96	0.98	0.95	0.91	0.98	0.95	0.87	0.90
	$K = 1$	1.13	1.08	1.11	1.02	1.03	1.10	1.02	1.00	1.09
	$K = 2$	1.29	1.21	1.24	1.54	1.18	1.35	1.42	1.13	1.96
	$K = 3$	1.46	1.33	1.37	5.05	1.99	2.91	4.30	1.67	10.05
	$K = 4$	1.63	1.46	1.50	10.40	5.78	9.12	9.76	4.88	24.99
$t = 2$	$K = 0$	0.92	0.92	0.96	0.93	0.88	0.96	0.94	0.83	0.88
	$K = 1$	1.08	1.04	1.09	0.97	1.00	1.08	0.98	0.97	1.04
	$K = 2$	1.24	1.16	1.22	1.04	1.06	1.19	1.04	1.05	1.24
	$K = 3$	1.40	1.28	1.35	1.53	1.18	1.66	1.40	1.15	2.39
	$K = 4$	1.57	1.40	1.47	4.69	1.60	4.00	3.92	1.41	8.56

²It worthwhile to mention that the second approach can be used just for inflated models.

Table 9: The rate premium for three categories A_1 , A_2 , and A_3 using well fitted models, whenever exact number of reported claim for each year of the policyholder's experience has been considered.

Year	Number of reported claims at year l (k_l)	Model:								
		$1INBM_1$			$1INBM_2$			$1INBM_3$		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$t = 0$	—	1	1	1	1	1	1	1	1	1
$t = 1$	$k_1 = 0$	0.64	0.63	0.88	0.83	0.81	0.96	0.79	0.63	0.96
	$k_1 = 1$	1.81	1.55	1.54	1.26	1.30	1.13	1.39	1.29	1.17
	$k_1 = 2$	6.52	3.91	4.86	2.50	2.43	2.55	3.55	2.54	2.21
	$k_1 = 3$	9.44	4.94	6.86	3.30	3.29	3.64	4.93	3.50	2.85
	$k_1 = 4$	12.37	6.38	8.85	4.13	4.12	4.54	6.31	4.46	3.49
$t = 2$	$k_1 = 0, k_2 = 0$	0.48	0.46	0.78	0.72	0.68	0.93	0.65	0.48	0.93
	$k_1 = 0, k_2 = 1$	1.15	1.03	1.33	1.06	1.05	1.09	1.10	0.84	1.03
	$k_1 = 0, k_2 = 2$	4.78	2.56	4.33	2.19	2.01	2.46	2.98	1.79	2.16
	$k_1 = 1, k_2 = 0$	1.15	1.03	1.33	1.06	1.05	1.09	1.10	0.84	1.13
	$k_1 = 1, k_2 = 1$	2.87	2.06	2.31	1.57	1.61	1.30	1.90	1.53	1.16
	$k_1 = 1, k_2 = 2$	6.79	3.58	5.63	2.76	2.60	2.82	3.93	2.47	2.38
	$k_1 = 2, k_2 = 0$	4.78	2.56	4.33	2.19	2.01	2.46	2.98	1.79	2.15
	$k_1 = 2, k_2 = 1$	6.79	3.58	5.63	2.76	2.60	2.82	3.93	2.47	2.38
	$k_1 = 2, k_2 = 2$	9.10	4.67	7.88	3.67	3.34	3.99	5.31	3.09	3.37

To illustrate a guideline to use result of Tables 8 and 9, suppose that *either* Negative Binomial with 2 mixture components, NBM_2 , *or* 1-Inflated Negative Binomial with 2 mixture components, $1INBM_2$, can be considered as an appropriate model. Now consider the following three different scenarios.

Scenario 1: For a given policyholder, no covariates information is available, category A_1 in Table 7. Based upon Table 8's and Table 9's result, respectively, his/her second year rate premium under NBM_2 model is 0.95 units while his/her second year rate premium under $1INBM_2$ model is 0.83 units, whenever such policyholder does not report any claim in the first year. But in the situation that such policyholder reports 2 claims in the first year. He/she has to pay 1.54 units, under NBM_2 model, and 2.50 units, under $1INBM_2$ model.

Scenario 2: The given policyholder belongs to category A_2 of Table 7. Based upon Table 8's and Table 9's result, respectively, his second year rate premium, under NBM_2 model, is 0.91 units while his second year rate premium, under $1INBM_2$ model, is 0.81 units, whenever such policyholder does not report any claim in the first year. But in the situation that such policyholder reports 2 claims in the first year. He has to pay 1.18 units, under NBM_2 model, and 2.43 units, under $1INBM_2$ model.

Scenario 3: The given policyholder belongs to category A_3 of Table 7. Based upon Table 8's and Table 9's result, respectively, her second year rate premium, under NBM_2 model, is 0.98 units while her second year rate premium, under $1INBM_2$ model, is 0.96 units, whenever such policyholder does not report any claim in the first year. But in the situation that such policyholder reports 2 claims in the first year. She has to pay 1.35 units, under NBM_2 model, and 2.55 units, under $1INBM_2$ model.

The above simple example, as well as other possible examples, shows that: **(1)** the inflated models and covariates information improve fairness of calculated rate premium; and **(2)** in the situation that number of reported claims uniformly distributed in past experience of a policyholder (for instance $k_1 = 1$ and $k_2 = 1$ instead of $k_1 = 0$ and $k_2 = 2$). His/Her rate premium under inflated models is more fair and acceptable.

Now, to estimate the pure premium, we consider one mixture Pareto distribution/regression model, as an appropriate model for claim's severity, along with other well fitted counting models. Moreover, we study situation that total claim size is *either* 1000 units (Case A) *or* 5000 unites (Case B). Table 10 and Table 11 show the pure premium under these assumptions.

Table 10: The pure premium for three categories A_1 , A_2 , and A_3 using well fitted models, whenever total claim size either 1000 or 5000 unites and exact number of reported claim for each year of the policyholder's experience has been considered.

Case A: Total of reported claim reach to 1000 unites										
Number of cumulated Year claims up to this year (K)		Model:								
		NBM_1 & Pareto M_1			NBM_2 & Pareto M_1			NBM_3 & Pareto M_1		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$t = 0$	—	613.739	723.848	461.546	607.554	730.992	463.354	623.147	726.489	449.722
$t = 1$	$K = 0$	589.189	694.894	452.315	574.376	667.716	453.895	594.470	634.635	405.854
	$K = 1$	629.269	723.241	460.357	561.898	650.758	441.088	576.836	631.235	423.590
	$K = 2$	622.519	707.323	448.918	735.309	652.794	470.045	697.730	620.015	661.912
	$K = 3$	621.617	689.808	440.058	2128.105	974.677	900.347	1860.741	811.248	3013.693
	$K = 4$	620.905	680.503	432.993	3921.243	2548.529	2535.831	3775.850	2137.017	6735.508
$t = 2$	$K = 0$	564.640	665.940	443.084	568.061	641.339	446.372	587.939	605.724	396.865
	$K = 1$	601.425	696.455	452.063	533.528	626.073	429.213	553.127	607.359	399.204
	$K = 2$	598.391	678.095	441.677	497.293	580.094	413.935	510.093	570.899	416.165
	$K = 3$	596.071	663.875	433.633	643.233	573.089	510.020	603.755	555.112	714.869
	$K = 4$	598.049	652.537	424.333	1769.746	702.016	1106.136	1514.878	613.404	2298.398
Case B: Total of reported claim reach to 5000 unites										
Number of cumulated Year claims up to this year (K)		Model:								
		NBM_1 & Pareto M_1			NBM_2 & Pareto M_1			NBM_3 & Pareto M_1		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$t = 0$	—	613.739	723.848	461.546	607.554	730.992	463.354	623.147	726.489	449.722
$t = 1$	$K = 0$	589.189	694.894	452.315	574.376	667.716	453.895	594.470	634.635	405.854
	$K = 1$	799.370	944.429	550.863	713.788	686.966	456.491	732.764	666.356	438.382
	$K = 2$	790.796	923.642	537.174	934.074	689.115	486.460	886.338	654.512	685.028
	$K = 3$	789.650	900.770	526.572	2703.365	1028.907	931.789	2363.729	856.385	3118.939
	$K = 4$	788.745	888.620	518.119	4981.215	2690.327	2624.388	4796.520	2255.920	6970.728
$t = 2$	$K = 0$	564.640	665.940	443.084	568.061	641.339	446.372	587.939	605.724	396.865
	$K = 1$	763.999	909.450	540.937	677.749	643.612	436.741	702.646	624.374	406.205
	$K = 2$	760.145	885.475	528.510	631.719	596.345	421.194	647.979	586.893	423.463
	$K = 3$	757.198	866.907	518.885	817.108	589.144	518.964	766.960	570.663	727.407
	$K = 4$	759.711	852.102	507.757	2248.135	721.683	1125.535	1924.372	630.588	2338.707

Table 11: The pure premium for three categories A_1 , A_2 , and A_3 using well fitted models, whenever total claim size either 1000 or 5000 unites and exact number of reported claim for each year of the policyholder's experience has been considered.

Case A: Total of reported claim reach to 1000 unites										
Year	Number of reported claims at year l (k_l)	Model:								
		1INBM ₁ & ParetoM ₁			1INBM ₂ & ParetoM ₁			1INBM ₃ & ParetoM ₁		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$t = 0$	–	623.391	757.191	571.461	611.317	733.685	534.525	609.443	790.408	566.132
$t = 1$	$k_1 = 0$	398.970	477.030	502.886	507.393	594.285	513.144	481.452	497.957	543.487
	$k_1 = 1$	1023.795	1085.798	790.796	698.893	882.399	542.755	768.625	943.307	595.197
	$k_1 = 2$	3195.859	2390.932	2178.476	1201.672	1439.796	1069.149	1701.115	1621.325	981.386
	$k_1 = 3$	4019.222	2562.144	2203.502	1405.025	1706.367	1169.205	2099.022	1815.284	915.449
	$k_1 = 4$	4712.021	2973.705	2554.659	1573.213	1920.324	1310.525	2403.626	2078.797	1007.430
$t = 2$	$k_1 = 0, k_2 = 0$	219.378	259.780	332.448	322.694	372.101	366.774	290.423	282.966	388.461
	$k_1 = 0, k_2 = 1$	650.477	721.531	682.960	587.958	712.707	523.542	608.265	614.247	523.977
	$k_1 = 0, k_2 = 2$	2342.976	1565.418	1940.906	1052.665	1190.942	1031.414	1427.979	1142.587	954.742
	$k_1 = 1, k_2 = 0$	650.477	721.531	682.960	587.958	712.707	523.542	608.265	614.247	523.977
	$k_1 = 1, k_2 = 1$	1406.766	1259.673	1035.449	754.650	953.939	545.056	910.456	976.625	559.523
	$k_1 = 1, k_2 = 2$	2936.409	1942.306	2239.077	1170.474	1366.822	1049.038	1661.517	1398.870	937.710
	$k_1 = 2, k_2 = 0$	2342.976	1565.418	1940.906	1052.665	1190.942	1031.414	1427.979	1142.587	954.742
	$k_1 = 2, k_2 = 1$	2936.409	1942.306	2239.077	1170.474	1366.822	1049.038	1661.517	1398.870	937.710
	$k_1 = 2, k_2 = 2$	3520.914	2276.944	2816.357	1392.471	1577.924	1333.877	2008.510	1572.678	1193.225
Case B: Total of reported claim reach to 5000 unites										
Year	Number of reported claims at year l (k_l)	Model:								
		1INBM ₁ & ParetoM ₁			1INBM ₂ & ParetoM ₁			1INBM ₃ & ParetoM ₁		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$t = 0$	–	623.391	757.191	571.461	611.317	733.685	534.525	609.443	790.408	566.132
$t = 1$	$k_1 = 0$	398.970	477.030	502.886	507.393	594.285	513.144	481.452	497.957	543.487
	$k_1 = 1$	1300.542	1417.866	946.265	887.815	1152.261	649.459	976.396	1231.797	712.211
	$k_1 = 2$	4059.749	3122.146	2606.761	1526.503	1880.126	1279.341	2160.953	2117.171	1174.325
	$k_1 = 3$	5105.682	3345.717	2636.704	1784.825	2228.221	1399.067	2666.421	2370.447	1095.424
	$k_1 = 4$	5985.752	3883.148	3056.902	1998.477	2507.613	1568.174	3053.363	2714.552	1205.490
$t = 2$	$k_1 = 0, k_2 = 0$	219.378	259.780	332.448	322.694	372.101	366.774	290.423	282.966	388.461
	$k_1 = 0, k_2 = 1$	826.311	942.195	817.229	746.892	930.673	626.469	772.688	802.100	626.989
	$k_1 = 0, k_2 = 2$	2976.319	2044.167	2322.484	1337.216	1555.166	1234.188	1813.983	1492.022	1142.443
	$k_1 = 1, k_2 = 0$	826.311	942.195	817.229	746.892	930.673	626.469	772.688	802.100	626.989
	$k_1 = 1, k_2 = 1$	178.7037	1644.916	1239.016	958.644	1245.680	652.213	1156.567	1275.304	669.525
	$k_1 = 1, k_2 = 2$	3730.166	2536.317	2679.275	1486.871	1784.835	1255.277	2110.651	1826.684	1122.062
	$k_1 = 2, k_2 = 0$	2976.319	2044.167	2322.484	1337.216	1555.166	1234.188	1813.983	1492.022	1142.443
	$k_1 = 2, k_2 = 1$	3730.166	253.6317	2679.275	148.6871	1784.835	1255.277	2110.651	1826.684	1122.062
	$k_1 = 2, k_2 = 2$	4472.672	2973.297	3370.048	1768.877	2060.497	1596.115	2551.441	2053.647	1427.811

Same as the above, to illustrate a guideline to use result of Tables 10 and 11, suppose that *either* Negative Binomial with 2 mixture components, NBM_2 , *or* 1-Inflated Negative Binomial with 2 mixture components, $1INBM_2$, can be considered as an appropriate model for claim frequency. Now consider the following three different scenarios.

Scenario 1: For a given policyholder in category A_1 of Table 7. Based upon Table 10's and Table 11's result, respectively, his/her second year pure premium under NBM_2 model is 622.519 units while his/her second year pure premium under $1INBM_2$ model is 3195.859 units, whenever such policyholder reported 2 claims with total size 1000 units in the first year. But in the situation that total size of two reported claims reach to 5000 units. He/she has to pay 790.796 units, under NBM_2 model, and 4059.749 units, under $1INBM_2$ model.

Scenario 2: The given policyholder belongs to category A_2 of Table 7. Based upon Table 10's and Table 11's result, respectively, his second year pure premium, under NBM_2 model, is 707.323 units while his second year pure premium, under $1INBM_2$ model, is 2390.932 units, whenever such policyholder reported 2 claims with total size 1000 units in the first year. But in the situation that total size of two reported claims reach to 5000 units. He has to pay 932.642 units, under NBM_2 model, and 3122.146 units, under $1INBM_2$ model.

Scenario 3: The given policyholder belongs to category A_3 of Table 7. Based upon Table 10's and Table 11's result, respectively, her second year pure premium, under NBM_2 model, is 440.918 units while her second year pure premium, under $1INBM_2$ model, is 2178.476 units, whenever such policyholder reported 2 claims with total size 1000 units in the first year. But in the situation that total size of two reported claims reach to 5000 units. She has to pay 537.174 units, under NBM_2 model, and 2606.761 units, under $1INBM_2$ model.

The above simple example shows that: (1) the inflated models provides more fair pure premium of policyholders who made some claims in their past experience. While for both cases A and B, the pure premium under non-inflated models do not fairly penalized such policyholders; and (2) in the

situation that number of reported claims uniformly distributed in past experience of a policyholder (for instance $k_1 = 1$ and $k_2 = 1$ instead of $k_1 = 0$ and $k_2 = 2$). His/Her pure premium under inflated models is more appealing and acceptable.

7. Conclusion and suggestion

This article introduces an k-Inflated Negative Binomial mixture (kIBNM) distribution/regression model and provides an EM algorithm to estimate its parameters. As an application of the kIBNM distribution/regression to model number of reported claim under a rate-making system has been given. Moreover, in order to compute the pure premium under the system, severity of reported claim has been model with a Pareto mixture distribution/regression model. As an application frequency of reported claim of Iranian third party liability, at 2011, has been model by the kIBNM and all of possible models that have been used by authors. Numerical illustration shows that: **(1)** the kIBNM models provide more fair rate/pure premiums for policyholders under a rate-making system; and **(2)** in the situation that number of reported claims uniformly distributed in past experience of a policyholder (for instance $k_1 = 1$ and $k_2 = 1$ instead of $k_1 = 0$ and $k_2 = 2$). The rate/pure premium under the kIBNM models are more appealing and acceptable.

We conjecture that the result of this article may be improved by considering a Double Inflated Negative Binomial with probability mass function $P(Y = y|\boldsymbol{\theta}) = p_1 I_{k_1}(y) + p_2 I_{k_2}(y) + \sum_{j=3}^m p_j \binom{y+\alpha_j-1}{y} \left(\frac{\tau_j}{\alpha_j+\tau_j}\right)^{\alpha_j} \left(\frac{\alpha_j}{\alpha_j+\tau_j}\right)^{\tau_j}$ where $k_1, k_2 \in \mathbb{N}$, $\sum_{j=1}^m p_j = 1$, and $p_j, \alpha_j, \tau_j \geq 0$, for all $j = 1, \dots, m$.

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